THE RELATION BETWEEN THE LOWERING OF THE PIEZOMETRIC SURFACE AND THE RATE AND DURATION OF DISCHARGE OF A WELL USING GROUND-WATER STORAGE

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When a well is pumped or otherwise discharged, water-levels in its neighborhood are lowered. Unless this lowering occurs instantaneously it represents a loss of storage, either by the unwatering of a portion of the previously saturated sediments if the aquifer is nonartesian or by release of stored water by the compaction of the aquifer due to the lowered pressure if the aquifer is artesian. The mathematical theory of ground-water hydraulics has been based, apparently entirely, on a postulate that equilibrium has been attained and therefore that water-levels are no longer falling. In a great number of hydrologic problems, involving a well or pumping district near or in which water-levels are falling, the current theory is therefore not strictly applicable. This paper investigates in part the nature and consequences of a mathematical theory that considers the motion of ground-water before equilibrium is reached and, as a consequence, involves time as a variable.

To the extent that Darcy's law governs the motion of ground-water under natural conditions and under the artificial conditions set up by pumping, an analogy exists between the hydrologic conditions in an aquifer and the thermal conditions in a similar thermal system. Darcy's law is analogous to the law of the flow of heat by conduction, hydraulic pressure being analogous to temperature, pressure-gradient to thermal gradient, permeability to thermal conductivity, and specific yield to specific heat. Therefore, the mathematical theory of heat-conduction developed by Fourier and subsequent writers is largely applicable to hydraulic theory. This analogy has been recognized, at least since the work of Slichter, but apparently no attempt has been made to introduce the function of time into the mathematics of ground-water hydrology. Among the many problems in heat-conduction analogous to those in ground-water hydraulics are those concerning sources and sinks, sources being analogous to recharging wells and sinks to ordinary discharging wells.

C. I. Lubin, of the University of Cincinnati, has with great kindness prepared for me the following derivation of the equation expressing changes in temperature due to the type of source or sink that is analogous to a recharging or discharging well under certain ideal conditions, to be discussed below.

The equation given by H. S. Carslaw (Introduction to the mathematical theory of the conduction of heat in solids, 2nd ed., p. 152, 1921) for the temperature at any point in an infinite plane with initial temperature zero at any time due to an "instantaneous line-source coinciding with the axis of z of strength Q" (involving two-dimensional flow of heat) is

$$v = (Q/4\pi\kappa t)\ e^{-(x^2 + y^2)/4\kappa t} \tag{1}$$

where v = change in temperature at the point x,y at the time t; Q = the strength of the source or sink--in other words, the amount of heat added or taken out instantaneously divided by the specific heat per unit-volume; κ = Kelvin's coefficient of diffusivity, which is equal to the coefficient of conductivity divided by the specific heat per unit-volume; and t = time.

The effect of a continuous source or sink of constant strength is derived from equation (1) as follows: Let $Q = \varphi(t')dt'$; then $v_{(x,y,t)} = \int_0^t [\varphi(t')/4\pi\kappa(t-t')]\ e^{-(x^2+y^2)/4\kappa(t-t')}\ dt'$. Let $\varphi(t') = \lambda$, a constant; then $v_{(t)} = (\lambda/4\pi\kappa)\int_0^t [e^{-(x^2+y^2)/4\kappa(t-t')}/(t-t')]\ dt'$. Let $u = (x^2+y^2)/4\kappa(t-t')$; then

$$v_{(t)} = (\lambda/4\pi\kappa)\int_{(x^2+y^2)/4\kappa t}^{\infty} [e^{-u}/(t-t')]\ [(x^2+y^2)/4\kappa]\ [du/u^2]$$

$$= (\lambda/4\pi\kappa)\int_{(x^2+y^2)/4\kappa t}^{\infty} (e^{-u}/u)du \tag{2}$$

The definite integral, $\int_{(x^2+y^2)/4\kappa t}^{\infty} (e^{-u}/u)du$, is a form of the exponential integral, tables of which are available (Smithsonian Physical Tables, 8th rev. ed., table 32, 1933; the values to be used are those given for Ei (-x), with the sign changed). The value of the integral is given by the series

$$\int_x^{\infty} (e^{-u}/u)du = -0.577216 - \log_e x + x - x^2/2\cdot 2! + x^3/3\cdot 3! - x^4/4\cdot 4! + \ldots\ldots \tag{3}$$

Equation (2) can be immediately adapted to ground-water hydraulics to express the draw-down at any point at any time due to pumping a well. The coefficient of diffusivity, , is analogous to the coefficient of transmissibility of the aquifer divided by the specific yield. (The term "coefficient of transmissibility" is here used to denote the product of Meinzer's coefficient of permeability and the thickness of the saturated portion of the aquifer; it quantitatively describes the ability of the aquifer to transmit water. Meinzer's coefficient of permeability denotes a characteristic of the material; the coefficient of transmissibility denotes the analogous characteristic of the aquifer as a whole.) The continuous strength of the sink is analogous to the pumping rate divided by the specific yield. Making these substitutions, we have

$$v = (F/4\pi\tau)\int_{r^2s/4\tau t}^{\infty} (e^{-u}/u)du \tag{4}$$

in which the symbols have the meanings given with equation (5). In equation (4) the same units must of course be used throughout. Equation (4) may be adapted to units commonly used

$$v = (114.6F/\tau)\int_{1.87r^2s/\tau t}^{\infty} (e^{-u}/u)du \tag{5}$$

where v = the draw-down, in feet, at any point in the vicinity of a well pumped at a uniform rate; F = the discharge of the well, in gallons a minute; τ = the coefficient of transmissibility of aquifer, in gallons a day, through each 1-foot strip extending the height of the aquifer, under a unit-gradient--this is the average coefficient of permeability (Meinzer) multiplied by the thickness of the aquifer; r = the distance from pumped well to point of observation, in feet; s = the specific yield, as a decimal fraction; and t = the time the well has been pumped, in days.

Equation (5) gives the draw-down at any point around a well being pumped uniformly (and continuously) from a homogeneous aquifer of constant thickness and infinite areal extent at any time. The introduction of the function, time, is the unique and valuable feature of the equation. Equation (5) reduces to Thiem's or Slichter's equation for artesian conditions when the time of pumping is large.

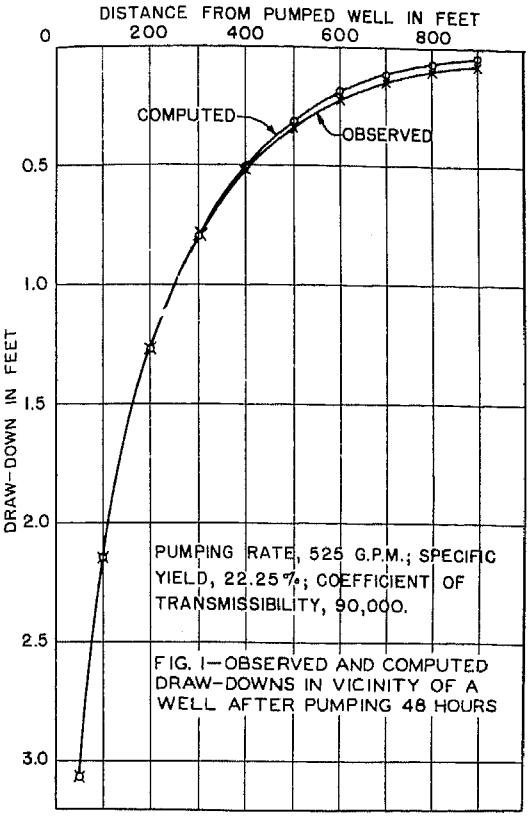

FIG. 1—OBSERVED AND COMPUTED DRAW-DOWNS IN VICINITY OF A WELL AFTER PUMPING 48 HOURS

Empirical tests of the equation are best made with the data obtained by L. K. Wenzel (Recent investigations of Thiem's method for determining permeability of water-bearing sediments, Trans. Amer. Geophys. Union, 13th annual meeting, pp. 313-317, 1932; also Specific yield determined from a Thiem's pumping test, Trans. Amer. Geophys. Union, 14th annual meeting, pp. 475-477, 1933) from pumping tests in the Platte Valley in Nebraska. Figure 1 presents the comparison of the computed and observed draw-downs after two days of pumping. The observed values are those of the generalized depression of the water-table as previously determined by Mr. Wenzel. The computed values are obtained by equation (5), using values of permeability and specific yield that are within one per cent of those determined by Mr. Wenzel by other methods. The agreement represented may be regarded as showing either that the draw-downs have been computed from known values of transmissibility and specific yield or that these factors have been computed from the known draw-downs.

Theoretically, the equation applies rigidly only to water-bodies (1) which are contained in entirely homogeneous sediments, (2) which have infinite areal extent, (3) in which the well penetrates the entire thickness of the water-body, (4) in which the coefficient of transmissibility is constant at all times and in all places, (5) in which the pumped well has an infinitesimal diameter, and (6) - applicable only to unconfined water-bodies - in which the water in the volume of sediments through which the water-table has fallen is discharged instantaneously with the fall of the water-table.

These theoretical restrictions have varying degrees of importance in practice. The effect of heterogeneity in the aquifer can hardly be foretold. The effect of boundaries can be considered by more elaborate analyses, once they are located. The effect of the well failing to penetrate the entire aquifer is apparently negligible in many cases. The pumped well used in the set-up that yielded the data for Figure 1 penetrated only 30 feet into a 90-foot aquifer. The coefficient of transmissibility must decrease during the process of pumping under water-table conditions, because of the diminution in the cross-section of the area of flow due to the fall of the water-table; however, it appears from Figure 1 that if the water-table falls through a distance equal only to a small percentage of the total thickness of the aquifer the errors are not large enough to be observed. In artesian aquifers the coefficient of transmissibility probably decreases because of the compaction of the aquifer, but data on this point are lacking. The error due to the finite diameter of the well is apparently always insignificant.

In heat-conduction a specific amount of heat is lost concomitantly and instantaneously with fall in temperature. It appears probable, analogously, that in elastic artesian aquifers a specific amount of water is discharged instantaneously from storage as the pressure falls. In nonartesian aquifers, however, the water from the sediments through which the water-table has fallen drains comparatively slowly. This time-lag in the discharge of the water made available from storage is neglected in the mathematical treatment here given. Hence an error is always present in the equation when it is applied to water-table conditions. However, inasmuch as the rate of fall of the water-table decreases progressively after a short initial period, it seems probable that as pumping continues the rate of drainage of the sediments tends to catch up with the rate of fall of the water-table, and hence that the error in the equation becomes progressively smaller.

For instance, although the draw-downs computed for a 24-hour period of pumping in Mr. Wenzel's test showed a definite lack of agreement with the observations, similar computations for a

48-hour period gave the excellent agreement shown in Figure 1. Unfortunately data for periods of pumping longer than 48 hours have not been available.

The equation implies that any two observations of draw-down, whether at different places or at the same place at different times, are sufficient to allow the computation of specific yield and transmissibility. However, more observations are always necessary in order to guard against the possibility that the computations will be vitiated by the heterogeneity of the aquifer. Moreover, it appears that the time-lag in the drainage of the unwatered sediments makes it impossible at present to compute transmissibility and specific yield from observations on water-levels in only one observation-well during short periods of pumping. Good data from artesian wells have not been available, but such data as we have hold out the hope that transmissibility and specific yield may be determined from data from only one observation-well.

A useful corollary to equation (5) may be derived from an analysis of the recovery of a pumped well. If a well is pumped for a known period and then left to recover, the residual draw-down at any instant will be the same as if pumping of the well had been continued but a recharge well with the same flow had been introduced at the same point at the instant pumping stopped. The residual draw-down at any instant will then be

$$v' = (114.6F/T)\left[\int_{1.87r^2s/Tt}^{\infty} (e^{-u}/u)du - \int_{1.87r^2s/Tt'}^{\infty} (e^{-u}/u)du\right. \qquad (6)$$

where t is the time since pumping started and t' is the time since pumping stopped.

In and very close to the well the quantity $(1.87r^2s/Tt')$ will be very small as soon as t' ceases to be small, because r is very small. In many problems ordinarily met in ground-water hydraulics, all but the first two terms of the series of equation (3) may be neglected, so that, if $Z = (1.87r^2s/Tt)$ and $Z' = (1.87r^2s/Tt')$ equation (6) may be approximately rewritten

$$v' = (114.6F/T)\left[-0.577 - \log_e Z + 0.577 + \log_e Z(t/t')\right] = (114.6F/T)\log_e(t/t')$$

Transposing and converting to common logarithms, we have

$$T = (264F/v')\log_{10}(t/t') \qquad (7)$$

This equation permits the computation of the coefficient of transmissibility of an aquifer from an observation of the rate of recovery of a pumped well.

Figure 2 shows a plot of observed recovery-curves. The ordinates are log (t/t'); the abscissas are the distances the water-table lies below its equilibrium-position. According to equation (7) the points should fall on a straight line passing through the origin. Curve A is a plot of the recovery of a well within 3 feet of the well pumped for Mr. Wenzel's test, previously mentioned. Most of the points lie on a straight line, but the line passes below the origin. This discrepancy is probably due to the fact that the water-table rises faster than the surrounding pores are filled. The coefficient of permeability computed from the equation is about 1200, against a probably correct figure of 1000. Curve B is plotted from data obtained from an artesian well near Salt Lake City. The points all fall according to theory.

Curve C shows the recovery of a well penetrating only the upper part of a nonartesian aquifer of comparatively low transmissibility. It departs markedly from a straight line. This curve probably follows equation (6), but it does not follow equation (7), for in this case $(1.87r^2s/Tt')$ is not small. Equation (6), involving r and s, neither of which may be known in practice, is not of practical value for the present purpose. Further empirical tests may show that it is feasible to project the curve to the origin, in the neighborhood of which $(1.87r^2s/Tt')$ becomes small, owing to the increase in t and t', and apply equation (7) to the extrapolated values so obtained in order to determine at least an approximate value of the transmissibility.

The paramount value of equation (5) apparently lies in the fact that it gives part of the theoretical background for predicting the future effects of a given pumping regimen upon the water-levels in a district that is primarily dependent on ground-water storage. Such districts may include many of those tapping extensive nonartesian bodies of ground-water. Figure 3 shows the vertical rate of fall of the water-level in an infinite aquifer, the water being all taken from storage. The curves are plotted for certain definite values of pumping rate, transmissibility, and specific yield, but by changing the scales either curve could be made applicable to any values set up.

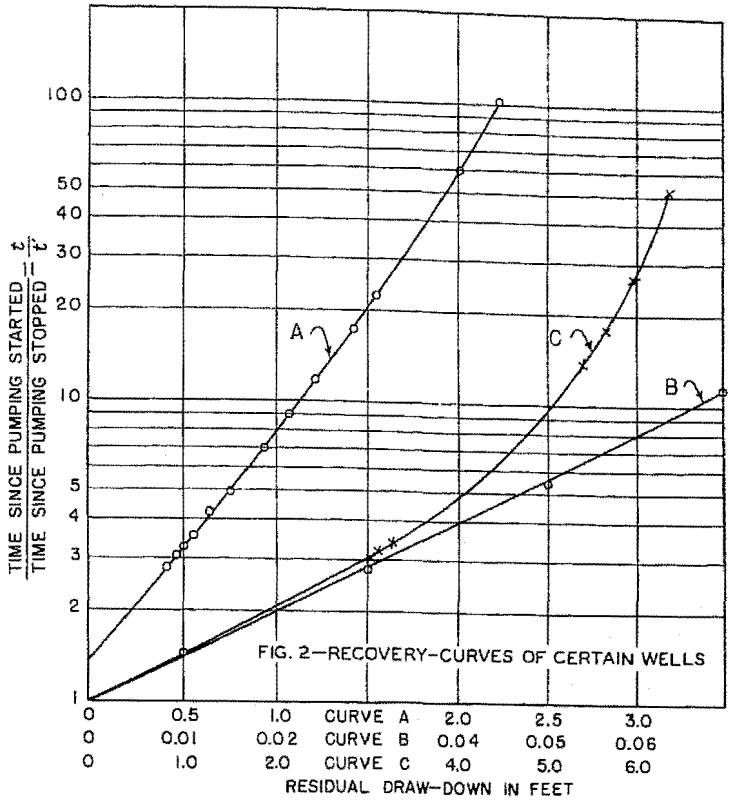

FIG. 2—RECOVERY-CURVES OF CERTAIN WELLS

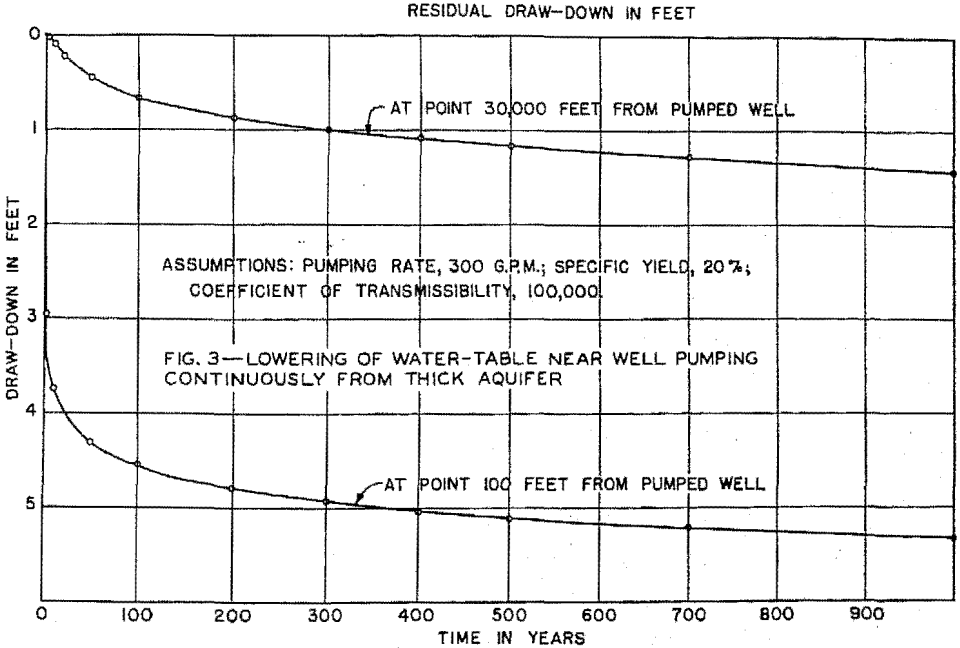

FIG. 3—LOWERING OF WATER-TABLE NEAR WELL PUMPING CONTINUOUSLY FROM THICK AQUIFER

These theoretical curves agree qualitatively with the facts generally observed when a well is pumped. The water-level close to the well at first falls very rapidly, but the rate of fall soon slackens. In the particular case considered in Figure 3 the water-level at a point 100 feet from the pumped well would fall during the first year of pumping more than half the distance it would fall in 1000 years. A delayed effect of the pumping is shown at distant points.

The water-level at a point about 6 miles from the pumped well of Figure 3 would fall only minutely for about five years but would then begin to fall perceptibly, although at a much less rate than the water-level close to the well. Incidentally the rate of fall after considerable pumping is so small that it might easily lead to a false assumption of equilibrium. The danger in a pumping district using ground-water storage lies in the delayed interference of the wells. For instance, although in 50 years one well would cause a draw-down of only 6 inches in another well 6 miles away, yet the 100 wells that might lie within 6 miles of a given well would cause in it a total draw-down of more than 50 feet.

In the preparation of this paper I have had the indispensable help not only of Dr. Lubin, who furnished the mathematical keystone of the paper, but also of Dr. C. E. Van Orstrand, of the United States Geological Survey, and of my colleagues of the Ground Water Division of the Survey, who cordially furnished data and criticism.

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